

A FORMULA FOR THE CONVERGENTS OF A CONTINUED FRACTION OF RAMANUJAN

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ABSTRACT. In Entry 16, Chapter 16 of his notebooks, Ramanujan himself gave a formula for the convergents of the famous Rogers-Ramanujan continued fraction. We provide a similar formula for the convergents of a more general continued fraction, namely Entry 15 of Chapter 16.

Keywords: Rogers-Ramanujan Continued Fraction, Ramanujan, the Lost Notebook.

1. THE THEOREM

The q -rising factorial $(a; q)_k$ is defined as $(a; q)_0 := 1$; and when $k > 0$, as the product of k terms:

$$(a; q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}).$$

Ramanujan wrote a ratio of two series as a continued fraction:

$$\frac{\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} \lambda^k}{\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q; q)_k} \lambda^k} = 1 + \frac{\lambda q}{1} + \frac{\lambda q^2}{1} + \frac{\lambda q^3}{1} + \cdots. \quad (1.1)$$

This is known as the Rogers-Ramanujan continued fraction (the two series on the left are the sum sides of the famous Rogers-Ramanujan identities!) and appears as a corollary to Entry 15 in Chapter 16 of Ramanujan's Notebooks (see Berndt [3]). Next, in Entry 16, Ramanujan provides a formula for the n th convergents of this continued fraction: For each positive integer n , let

$$\mu = \mu_n(\lambda, q) = \sum_{k=0}^{[(n+1)/2]} \frac{q^{k^2} \lambda^k}{(q; q)_k} \frac{(q; q)_{n-k+1}}{(q; q)_{n-2k+1}}$$

and

$$\nu = \nu_n(\lambda, q) = \sum_{k=0}^{[n/2]} \frac{q^{k^2+k} \lambda^k}{(q; q)_k} \frac{(q; q)_{n-k}}{(q; q)_{n-2k}}.$$

Then,

$$\frac{\mu}{\nu} = 1 + \frac{\lambda q}{1} + \frac{\lambda q^2}{1} + \frac{\lambda q^3}{1} + \cdots + \frac{\lambda q^n}{1}. \quad (1.2)$$

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Ramanujan's generalization of the Rogers-Ramanujan continued fraction is given by Entry 15 of Chapter 16 of [3]: For $|q| < 1$,

$$\frac{\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k (-bq; q)_k} \lambda^k}{\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q; q)_k (-bq; q)_k} \lambda^k} = 1 + \frac{\lambda q}{1 + bq} + \frac{\lambda q^2}{1 + bq^2} + \frac{\lambda q^3}{1 + bq^3} + \dots \quad (1.3)$$

where we have renamed some symbols in order to fit the notation used by Andrews and Berndt [2, Entry 6.3.1 (ii)].

However, Ramanujan did not provide a formula for the convergents of this continued fraction. Our objective here is to provide just such a formula, which is a natural extension of Ramanujan's own formula (1.2).

Theorem 1.4. *Let*

$$g_n(s) := \sum_{k=0}^{\infty} \frac{q^{k^2+sk} \lambda^k}{(q; q)_k (-bq^s; q)_k} \frac{(q; q)_{n-k-s+1}}{(q; q)_{n-2k-s+1}} \frac{(-bq; q)_{n-k}}{(-bq; q)_n}. \quad (1.5)$$

Then, for $n = 1, 2, 3, \dots$, we have

$$(1+b) \frac{g_n(0)}{g_n(1)} = 1 + b + \frac{\lambda q}{1 + bq} + \frac{\lambda q^2}{1 + bq^2} + \frac{\lambda q^3}{1 + bq^3} + \dots + \frac{\lambda q^n}{1 + bq^n}. \quad (1.6)$$

Observe that when $b = 0$, then (1.6) reduces to (1.2). In this case, $g_n(0)$ and $g_n(1)$ reduce to μ and ν respectively.

To obtain a formula for the convergents of (1.3), consider:

$$(1+b) \frac{g_n(0)}{g_n(1)} - b = 1 + \frac{\lambda q}{1 + bq} + \frac{\lambda q^2}{1 + bq^2} + \frac{\lambda q^3}{1 + bq^3} + \dots + \frac{\lambda q^n}{1 + bq^n}.$$

2. A PROOF

Before heading into the proof of (1.6), we need the definition of q -rising factorials, when k is not a non-negative integer. For that we need the infinite q -rising factorial

$$(a; q)_{\infty} := \prod_{j=0}^{\infty} (1 - aq^j), \text{ for } |q| < 1.$$

When k is not a positive integer, one can define

$$(a; q)_k = \frac{(a; q)_{\infty}}{(aq^k; q)_{\infty}}.$$

Observe that this definition implies

$$\frac{1}{(q; q)_m} = 0 \text{ when } m = -1, -2, -3, \dots$$

To prove (1.6), we use the approach used by Euler [5] (as explained by Bhatnagar [4]). We use the elementary identity:

$$\frac{N}{D} = 1 + \frac{N - D}{D} \quad (2.1)$$

to 'divide' two series.

Proof. Observe first that in the sum $g_n(s)$ in (1.5), the index k goes from 0 to $\lfloor \frac{n-s+1}{2} \rfloor$. Further, observe that

$$g_n(n) = 1 = g_n(n+1), \quad (2.2)$$

since only the terms corresponding to the index $k = 0$ survive.

We will show

$$(1 + bq^s) \frac{g_n(s)}{g_n(s+1)} = 1 + bq^s + \frac{\lambda q^{s+1}}{(1 + bq^{s+1}) \frac{g_n(s+1)}{g_n(s+2)}}, \quad (2.3)$$

for $s = 0, 1, 2, 3, \dots, n-1$. The formula (1.6) follows immediately by iterating (2.3) n times.

To prove (2.3), we use (2.1) to find that

$$(1 + bq^s) \frac{g_n(s)}{g_n(s+1)} = (1 + bq^s) \left(1 + \frac{g_n(s) - g_n(s+1)}{g_n(s+1)} \right). \quad (2.4)$$

Consider the difference of sums $g_n(s) - g_n(s+1)$.

$$\begin{aligned} g_n(s) - g_n(s+1) &= \sum_{k=0}^{\infty} \left(\frac{q^{k^2+sk} \lambda^k}{(q; q)_k (-bq^s; q)_{k+1}} \frac{(q; q)_{n-k-s}}{(q; q)_{n-2k-s+1}} \frac{(-bq; q)_{n-k}}{(-bq; q)_n} \right. \\ &\quad \times \left. \left[(1 + bq^{s+k})(1 - q^{n-k-s+1}) - (1 + bq^s)(1 - q^{n-2k-s+1})q^k \right] \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{q^{k^2+sk} \lambda^k}{(q; q)_k (-bq^s; q)_{k+1}} \frac{(q; q)_{n-k-s}}{(q; q)_{n-2k-s+1}} \frac{(-bq; q)_{n-k}}{(-bq; q)_n} \right. \\ &\quad \times \left. \left[(1 + bq^{n-k+1})(1 - q^k) \right] \right) \\ &= \sum_{k=1}^{\infty} \frac{q^{k^2+sk} \lambda^k}{(q; q)_{k-1} (-bq^s; q)_{k+1}} \frac{(q; q)_{n-k-s}}{(q; q)_{n-2k-s+1}} \frac{(-bq; q)_{n-k+1}}{(-bq; q)_n} \\ &= \sum_{k=0}^{\infty} \frac{q^{(k+1)^2+s(k+1)} \lambda^{k+1}}{(q; q)_k (-bq^s; q)_{k+2}} \frac{(q; q)_{n-k-s-1}}{(q; q)_{n-2k-s-1}} \frac{(-bq; q)_{n-k}}{(-bq; q)_n} \\ &= \frac{\lambda q^{s+1}}{(1 + bq^s)(1 + bq^{s+1})} g_n(s+2). \end{aligned}$$

Now by substituting in (2.4), we immediately obtain (2.3), and our proof is complete. \blacksquare

Notice that Ramanujan's continued fraction (1.3) is an immediate corollary of our formula. Take the limits $n \rightarrow \infty$ in (1.6). The continued fraction in (1.3) converges if its convergents converge, and we have found the convergents to be $(1+b)g_n(0)/g_n(1) - 1$ which converges when $|q| < 1$. (Getting the left hand side of (1.3) from this is a pleasant exercise. Try it!)

3. SOME CONNECTIONS

Ramanujan's Entry 16 (equation (1.2)) was rediscovered by P. Kesava Menon [10], and Hirschhorn [6]. Another formula for the convergents of

an even more general continued fraction of Ramanujan has been given by Hirschhorn [7, eq. (1)]. On taking $a = 0$ in Hirschhorn's formula, we obtain a formula (different from (1.6)) for the convergents of (1.3), where the numerator and denominator are double sums. Similar results appear in Hirschhorn [8] and [9].

There is also a sequence of orthogonal polynomials due to Al-Salam and Ismail [1], namely

$$U_n(x; a, b) = \sum_{k \geq 0} \frac{(-a; q)_{n-k} (q; q)_{n-k}}{(-a; q)_k (q; q)_k (q; q)_{n-2k}} x^{n-2k} (-b)^k q^{k(k-1)}.$$

This is similar to our $g_n(s)$. Indeed, we can see that

$$g_n(1) = \frac{1}{(-bq; q)_n} U_n(1; bq, -\lambda q^2).$$

Al-Salam and Ismail [1] have not explicitly stated (1.6), but perhaps our formula can be extracted from their work.

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